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UNSTEADY HEAT AND FLUID FLOW THROUGH A CURVED RECTANGULAR DUCT OF LARGE ASPECT RATIO

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ABSTRACT
In this paper, a comprehensive numerical study is presented for the fully developed two-dimensional flow of viscous incompressible fluid through a curved rectangular duct of aspect ratio 4. Unsteady flow characteristics are studied over a wide range of the Grashof number $100 \le Gr \le 2000$ for two cases of the Dean numbers, *Case I:* $Dn = 100$ and *Case II:* $Dn = 500$. Spectral method is used as a basic tool to solve the system of nonlinear differential equations. In order to investigate the nonlinear behavior of the unsteady solutions, time evolution calculations of the resistance coefficient are obtained. The main concern of the present study is to find out the unsteady flow characteristics that is whether the flow is steady-state, periodic multi-periodic or chaotic solutions, if the Dean number or the Grashof number is increased. It is found that the steady flow turns into chaotic flow through periodic and multi-periodic flows if the Grashof number is increased keeping the Dean number fixed. We also obtained secondary flow patterns and temperature profiles as the parameters are changed. It is found that the number secondary vortex decreases with the increase of the Grashof number at small Dean numbers, but the secondary vortex becomes two- to ten-vortex solutions if both the parameters are increased simultaneously.

Keywords: Curved Rectangular Duct, Secondary Flow, Time Evolution, Periodic Solution, Chaos.

1. INTRODUCTION
The study of flows through curved ducts and channels has been and continuous to be an area of paramount interest of many researchers because of the diversity of their practical applications in fluids engineering, such as in fluid transportation, turbo machinery, refrigeration, air conditioning systems, heat exchangers, chemical reactors, ventilators, centrifugal pumps, internal combustion engines and blade- to-blade passage for cooling system in modern gas turbines. Blood flow in the human and other animals also represents an important application of this subject because of the curvatures of many blood vessels, particularly the aorta.

Considering the non-linear nature of the Navier-Stokes equation, the existence of multiple solutions does not come as a surprise. However, an early complete bifurcation study of fully developed flows in a curved duct was conducted by Winters (1987). Yanase *et al.,* (2005) performed numerical investigation of isothermal and non-isothermal flows through a curved duct of rectangular cross-section. Mondal *et al.* (2006) performed numerical prediction of non-isothermal flows through a curved square duct over a wide range of the curvature and the Dean number. Recently, Mondal *et al.* (2007) numerically investigated the bifurcation diagram for two-dimensional steady flow through a curved square duct. Very recently, Mondal *et al.* (2009, 2010) performed bifurcation structure of the steady solutions and investigated linear stability of the solutions for the flow through a curved rectangular duct of small aspect ratio.

Time dependent analysis of fully developed curved duct flows was initiated by Yanase and Nishiyama (1988) for a rectangular cross section. Mondal *et al.* (2007) performed numerical prediction of the solution structure, stability and transitions of isothermal flow through a curved square duct. They showed that there is a close relationship between unsteady solutions and the bifurcation diagram of steady solutions. To the best of the authors' knowledge, however, there has not yet been done any substantial work studying the effects of large aspect ratio on unsteady solutions through a curved rectangular duct flows. This paper is, therefore, an attempt, to fill up this gap with a view to study the non-linear nature of the unsteady solutions for large aspect ratio, because this type of flow of often encountered in engineering applications.

In the present study, a numerical result is presented for the fully developed two-dimensional flow of viscous incompressible fluid through a curved rectangular duct. The main objective of the present study is to investigate the unsteady flow through a curved rectangular channel in the presence of buoyancy effect.

2. MATHEMATICAL FORMULATION
Consider a viscous incompressible fluid streaming through a curved duct with rectangular cross-sections.

The coordinate system with relevant notations is shown in Figure 1. It assumed that the flow is uniform in the z-direction which is driven by a constant pressure gradient G along the centre of the duct. It is assumed that the outer wall of the duct is heated while the inner wall cooled. The temperature of the outer wall is $T_0 + \Delta T$ and that of the inner wall is $T_0 - \Delta T$, where $\Delta T > 0$. *u*, *v* and *w* are the velocity components in the *x*-, *y*- and *z*-directions, respectively. The variables are non-dimensionalized by using the representative length and the representative velocity.

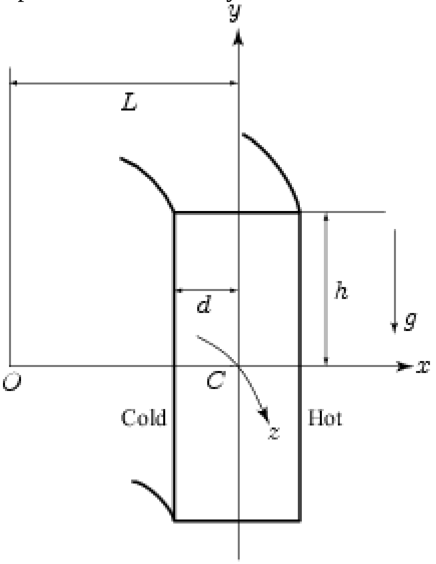


Fig 1. Coordinate system of the curved rectangular duct.

The sectional stream function $\psi(x, y)$ is introduced in the *x*- and *y*- directions as

$$u = \frac{1}{1+\delta x}\frac{\partial \psi}{\partial y}, \qquad v = -\frac{1}{1+\delta x}\frac{\partial \psi}{\partial x} \tag{1}$$

Then the basic equations for *w*, ψ and *T* are derived from the Navier-Stockes equations as

$$\begin{aligned}(1+\delta x)\frac{\partial w}{\partial t} + \frac{1}{l}\frac{\partial (w,\psi)}{\partial (x,y)} - Dn + \frac{\delta^2 w}{1+\delta x} = \\ (1+\delta x)\Delta_2 w - \frac{\delta}{l(1+\delta x)}\frac{\partial \psi}{\partial y} w + \delta \frac{\partial w}{\partial x}\end{aligned} \tag{2}$$

$$\begin{aligned}\left(\Delta_2 - \frac{\delta}{1+\delta x}\frac{\partial}{\partial x}\right)\frac{\partial \psi}{\partial t} = -\frac{1}{l(1+\delta x)}\frac{\partial (\Delta_2 \psi, \psi)}{\partial (x,y)} \\ + \frac{\delta}{l(1+\delta x)^2}\left[\frac{\partial \psi}{\partial y}\left(2\Delta_2 \psi - \frac{3\delta}{1+\delta x}\frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial x^2}\right)\right. \\ \left. - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x \partial y}\right] + \frac{\delta}{(1+\delta x)^2}\left[3\delta \frac{\partial^2 \psi}{\partial x^2} - \frac{3\delta^2}{1+\delta x}\frac{\partial \psi}{\partial x}\right] \\ - \frac{2\delta}{1+\delta x}\frac{\partial}{\partial x}\Delta_2 \psi + \frac{1}{l} w \frac{\partial \psi}{\partial y} - G_r\,(1+\delta x)\frac{\partial T}{\partial x} + \Delta_2^2 \psi\end{aligned} \tag{3}$$

$$\frac{\partial T}{\partial t} + \frac{1}{(1+\delta x)}\frac{\partial (T,\psi)}{\partial (x,y)} = \frac{1}{\Pr}\left(\Delta_2 T + \frac{\delta}{1+\delta x}\frac{\partial T}{\partial x}\right) \tag{4}$$

where

$$\Delta_2 \equiv \frac{\partial^2}{\partial x^2} + \frac{1}{l^2}\frac{\partial^2}{\partial y^2}, \qquad \frac{\partial (f,g)}{\partial (x,y)} \equiv \frac{\partial f \partial g}{\partial x \partial y} - \frac{\partial f \partial g}{\partial y \partial x}$$

The non-dimensional parameters Dn, the Dean number, Gr, the Grashof number and $\Pr$, the Prandtl number, which appear in equations (2) to (4) are defined as:

$$Dn = \frac{Gl^3}{\mu \upsilon}\sqrt{\frac{2l}{L}}, \quad Gr = \frac{\beta g \Delta T\, l^3}{\upsilon^2}, \quad \Pr = \frac{\upsilon}{\kappa}.$$

Here, *l* is the aspect ratio defined as $l = \frac{h}{d}$ and δ is the curvature.

The boundary conditions for w and ψ are used as

$$\begin{aligned} w(\pm 1, y) = w(x, \pm 1) = \psi(\pm 1, y) = \\ \psi(x, \pm 1) = \frac{\partial \psi}{\partial x}(\pm 1, y) = \frac{\partial \psi}{\partial y}(x, \pm 1) = 0 \end{aligned} \tag{5}$$

and the temperature T is assumed to be constant on the walls as:

$$T(1, y) = 1,\ T(-1, y) = -1, T(x, \pm 1) = x \tag{6}$$

3. NUMERICAL CALCULATION

In order to obtain the numerical solutions, spectral method is used. The main objective of the method is to use the expansion of the polynomial functions that is the variables are expanded in the series of functions consisting of Chebyshev polynomials. The expansion function $\phi_n(x)$ and $\psi_n(x)$ are expressed as

$$\begin{aligned}\phi_n(x) &= (1 - x^2) C_n(x), \\ \psi_n(x) &= (1 - x^2)^2 C_n(x)\end{aligned} \tag{7}$$

where $C_n(x) = \cos(n \cos^{-1}(x))$ is the n^{th} order Chebyshev polynomial. $w(x, y, t), \psi(x, y, t)$ and $T(x, y, t)$ are expanded in terms of the expansion functions $\phi_n(x)$ and $\psi_n(x)$ as:

$$\left.\begin{aligned} w(x, y, t) &= \sum_{m=0}^{M}\sum_{n=0}^{N} w_{mn}(t)\varphi_m(x)\varphi_n(y) \\ \psi(x, y, t) &= \sum_{m=0}^{M}\sum_{n=0}^{N} \psi_{mn}(t)\psi_m(x)\psi_n(y). \\ T(x,, y, t) &= \sum_{m=0}^{M}\sum_{n=0}^{N} T_{mn}\,\varphi_m(x)\varphi_n(y) + x \end{aligned}\right\} \tag{8}$$

where M and N are the truncation numbers in the x and y directions respectively. Unsteady solutions are obtained by using Crank-Nicolson and Adams-Bashforth methods together with the function expansion and collocation methods.

4. RESISTANT COEFFICIENT

The resistant coefficient λ is used as the representative quantity of the flow state and is generally used in fluids engineering, defined as

$$\frac{P_1^* - P_{21}^*}{\Delta_{z^*}} = \frac{\lambda}{d_h^*}\frac{1}{2}\rho\langle\omega^*\rangle^2 \tag{9}$$

where quantities with an P_1^* be asterisk denote dimensional ones, $\langle\ \rangle$ stands for the mean over the cross section of the duct and $d_h^* = 4(2d \times 2dl)/(4d + 4dl)$ is the hydraulic diameter. The main axial velocity $\langle\omega^*\rangle$ is calculated by

$$\langle\omega^*\rangle = \frac{\nu}{4\sqrt{2\delta}d}\int_{-1}^{1}dx\int_{-1}^{1}\omega(x, y, t)\,dy \tag{10}$$

Since $(P_1^* - P_2^*)/\Delta_{z^*} = G$, λ is related to the mean non-dimensional axial velocity $\langle\omega\rangle$ as

$$\lambda = \frac{8l\sqrt{2\delta}Dn}{(1+l)\langle\omega\rangle^2} \tag{11}$$

where $\langle\omega\rangle = \sqrt{2\delta}d\langle\omega^*\rangle/\nu$.

5. RESULTS AND DISCUSSION

5.1 *Case I*: Dn=100

In order to investigate the non-linear behavior of the unsteady solutions, time-evolution calculation of the resistance coefficient λ is performed for the curved rectangular duct of aspect ratio 4. Time evolution of λ for $Dn = 100$ and Gr = 100, 500, 1000, 1500 and 2000 is shown in Fig. 2, where it is seen that the flow is steady-state for Gr = 100, periodic at Gr = 500 and multi-periodic at Gr = 1000, 1500 and 2000. To show the multi-periodic oscillation more clearly, we show time evolution of λ for $Dn = 100$ and Gr =1000 in Fig. 3(a), where we see the flow is multi-periodic. To justify whether the flow is purely multi-periodic, we draw phase spaces of the time change of λ as shown in Fig. 3(b), where multi-periodic orbit is seen. Then we draw some contours of secondary flow patterns and temperature distributions in Fig. 4, where we observe that the unsteady flow at $Dn = 100$ and Gr =1000 oscillates between asymmetric two-vortex solutions.

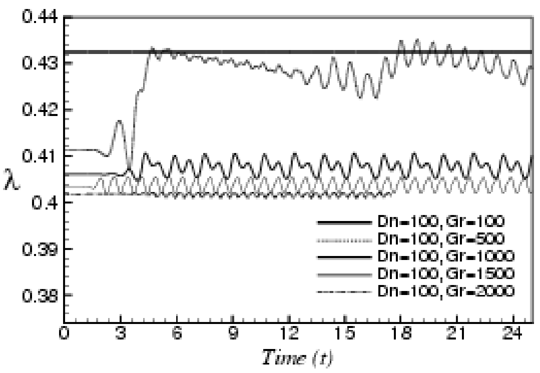
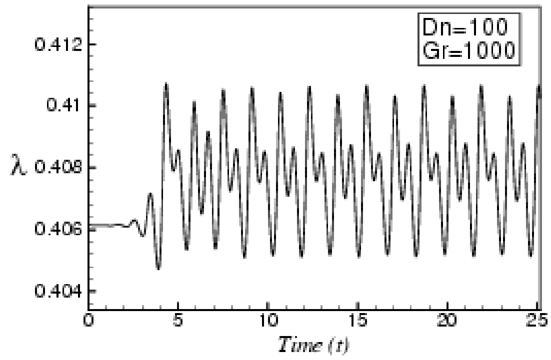
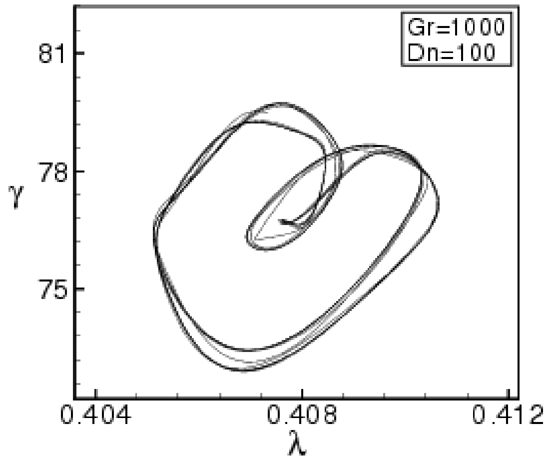


Fig 2. Time evolution of λ for the unsteady solutions at Dn =100 and $Gr = 100, 500, 1000, 1500, 2000$ for the aspect ratio 4.



(a)



(b)

Fig 3. (a) Time evolution of λ for the unsteady solutions at Dn =100 and $Gr = 1000$ for aspect ratio 4. (b) Phase plot in the $\lambda - \gamma$ plane, where $\gamma = \iint \psi\, dxdy$

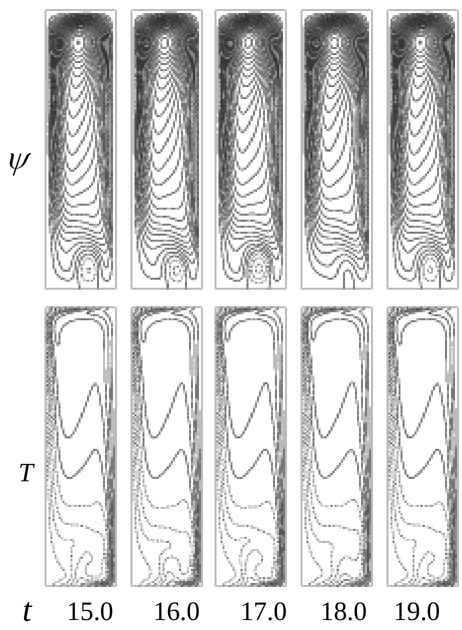


Fig 4. Contours of secondary flow and temperature profiles for $Dn = 100$ and $Gr = 1000$ at aspect ratio 4.

5.2 *Case II*: Dn=500

Then we investigate the non-linear behavior of the unsteady solutions by time-evolution calculation of the resistance coefficient λ for the curved rectangular duct of aspect ratio 4 at Dn =500 for various values of the Grashof numbers. Time evolution of λ for $Dn = 500$ and Gr = 100, 500, 1000, 1500 and 2000 is shown in Fig. 5, where it is found that the flow is chaotic at any value of Gr for Dn = 500. To show the chaotic oscillation more clearly, we show time evolution of λ for $Dn = 500$ and Gr =1000 in Fig. 6(a), where we see the flow is chaotic. To justify whether the flow is purely chaotic, we draw phase spaces of the time change of λ for Dn = 500 and Gr = 1000, for example, as shown in Fig. 6(b), where the chaotic orbit is seen. Then we draw some contours of secondary flow patterns and temperature distributions in Fig. 7 at Dn = 500 and Gr = 1000, where we observe that the unsteady flow at $Dn = 100$ and Gr =1000 is multi-vortex solution. Temperature distribution is consistent with secondary vortices.

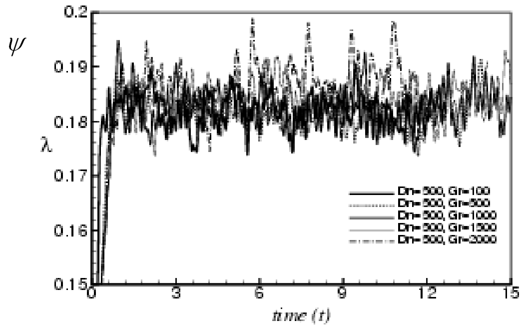
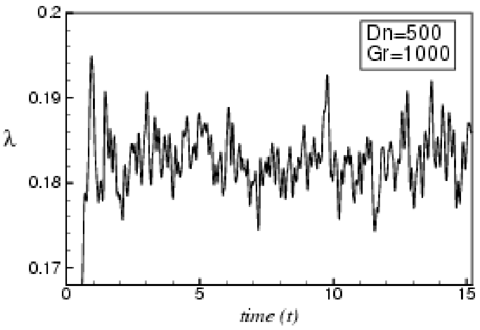
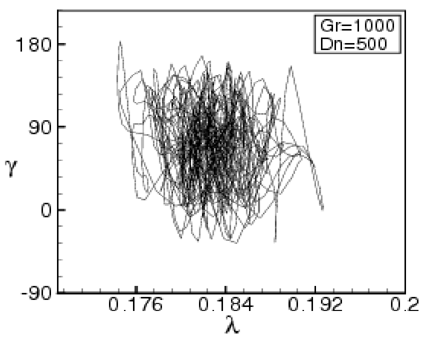


Fig 5. Time evolution of λ for the unsteady solutions at Dn =500 and $Gr = 100, 500, 1000, 1500, 2000$ for the aspect ratio 4.



(a)



(b)

Fig 6. (a) Time evolution of λ for the unsteady solutions at Dn =500 and $Gr = 1000$ for aspect ratio 4. (b) Phase plot in the $\lambda - \gamma$ plane, where $\gamma = \iint \psi \, dxdy$.

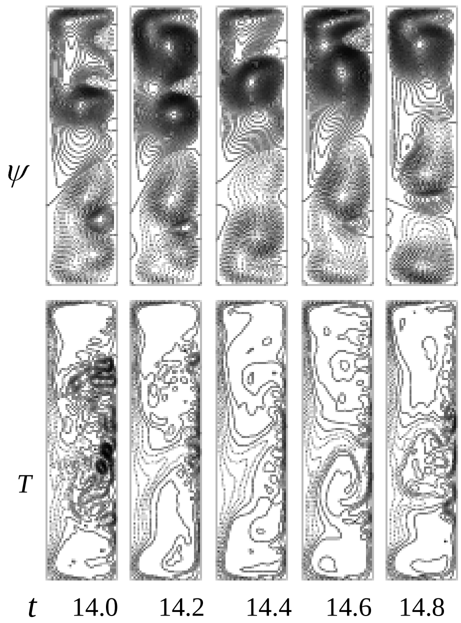


Fig 7. Contours of secondary flow and temperature profiles for $Dn = 500$ and $Gr = 1000$ for the aspect ratio 4.

Heat Transfer

In order to study the convective heat transfer from the heated wall to the fluid, the Nusselt number, *Nu*, is calculated for both the heated and cooled sidewalls. If the flow field is not steady, time-average of the Nusselt number, Nu_τ , is calculated. In Fig. 8, we show variation of the steady values of the Nusselt number with the Dean number for Gr = 500 and the aspect ratio 4, where a thick solid line denotes Nu_c on the inner (cooled) sidewall and a thin solid line Nu_h on the outer (heated) sidewall. Time-average of the Nusselt number, obtained by the time evolution computation of the Nusselt number for the heated and cooled sidewalls, is calculated at several values of the Dean number for both the periodic and chaotic solutions and plotted with the steady values of the Nusselt number in Fig. 8. It is found that time-averaged values of the Nusselt number are larger than the steady values of the Nusselt number for both the heated and cooled sidewalls, which suggests that occurrence of periodic or chaotic flow enhances heat transfer in the flow. It should be noted that the tendency of increasing the Nusselt number is larger on the heated sidewall than that on the cooled sidewall for larger Dean numbers i.e. where chaotic solutions occur, which can be explained by the fact that chaotic flow enhances heat transfer more frequently than the periodic solutions.

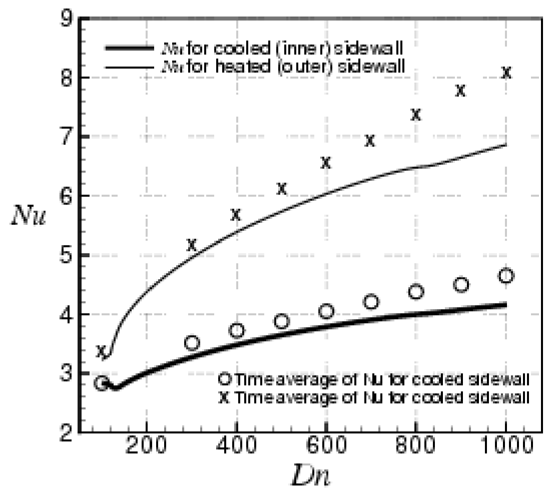


Fig 8. Variation of the Nusselt number (*Nu*) with the Dean number (*Dn*) for the steady values with time average of *Nu* for Gr = 500 and aspect ratio 4 (thick solid line: *Nu* for cooled sidewall, thin solid line: *Nu* for heated sidewall, O: time average of *Nu* for cooled sidewall, ×: time average of *Nu* for heated sidewall).

6. CONCLUSIONS

In this study, we investigate non-linear behavior of the unsteady solutions by time-evolution calculations for the flow through a curved rectangular duct of aspect ratio 4 for two cases of the Dean numbers, Case I: Dn =100 and Case II: Dn=500 over a wide range of the Grashof numbers. Time evolution calculation of the unsteady solutions for the aspect ratio 4 at Dn = 100 and at Gr = 100, 500, 1000, 1500 and 2000 shows that the flow is steady-state for Gr=100 but periodic or multi-periodic for Gr = 500, 1000, 1500 and 2000, which oscillates between asymmetric two-vortex solutions. Then we studied time evolution of the unsteady solutions for Dn = 500 at various values of Gr, and it is found that the flow is chaotic for all values of Gr investigated in this study. To justify whether the flow is periodic, multi-periodic or chaotic, phase space was found to be fruitful. Secondary flow patterns and temperature distributions are obtained at various parameters, and we obtained two- to ten-vortex solution. It is found that chaotic solution becomes strong at large *Dn*, which enhances heat transfer more frequently than the periodic states.

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